

# Rotation set for maps of degree 1 on the graph sigma

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December 7, 2007

## Abstract

For a continuous map on a topological graph containing a unique loop  $S$  it is possible to define the degree and, for a map of degree 1, rotation numbers. It is known that the set of rotation numbers of points in  $S$  is a compact interval and for every rational  $r$  in this interval there exists a periodic point of rotation number  $r$ . The whole rotation set (i.e. the set of all rotation numbers) may not be connected and it is not known in general whether it is closed.

The graph sigma is the space consisting in an interval attached by one of its endpoints to a circle. We show that, for a map of degree 1 on the graph sigma, the rotation set is closed and has finitely many connected components. Moreover, for all rational numbers  $r$  in the rotation set, there exists a periodic point of rotation number  $r$ .

## 1 Introduction

In [2] a rotation theory is developed for continuous self maps of degree 1 of topological graphs having a unique loop, using the ideas and techniques of [4, 3]. A rotation theory is usually developed in the universal covering space by using the liftings of the maps under consideration. The universal covering of a graph containing a unique loop is an “infinite tree modulo 1” (see Figure 1). It turns out that the rotation theory on the universal covering of a graph with a unique loop can be easily extended to the setting of infinite graphs that look like the space  $\hat{G}$  from Figure 2. These spaces are defined in detail in Section 2.1 and called *lifted graphs*. Each lifted graph  $T$  has a subset  $\hat{T}$  homeomorphic to the real line  $\mathbb{R}$  that corresponds to an “unwinding” of a distinguished loop of the original graph. In the sequel, we identify  $\hat{T}$  with  $\mathbb{R}$ .



Figure 1:  $G$  is the graph  $\sigma$ , its universal covering is  $T$ .

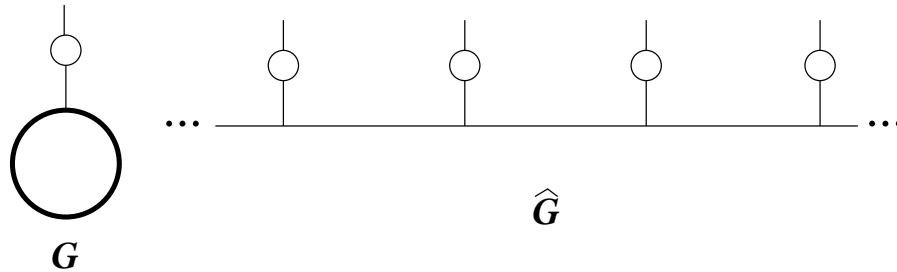


Figure 2: The graph  $G$  is unwound with respect to the bold loop to obtain  $\hat{G}$ , which is a lifted graph.

Given a lifted graph  $T$  and a map  $F$  from  $T$  to itself of degree one, there is no difficulty to extend the definition of rotation number to this setting in such a way that every periodic point still has a rational rotation number as in the circle case. However, the obtained rotation set  $\text{Rot}(F)$  may not be connected. Despite of this fact, it is proven in [2] that the set  $\text{Rot}_{\mathbb{R}}(F)$  corresponding to the rotation numbers of all points belonging to  $\mathbb{R}$ , has properties which are similar to (although weaker than) those of the rotation interval for a circle map of degree one. Indeed, this set is a compact non empty interval, if  $p/q \in \text{Rot}_{\mathbb{R}}(F)$  then there exists a periodic point of rotation number  $p/q$ , and if  $p/q \in \text{Int}(\text{Rot}_{\mathbb{R}}(F))$  then for all large enough positive integers  $n$  there exists a periodic point of period  $nq$  of rotation number  $p/q$ .

We conjecture that the whole rotation set  $\text{Rot}(F)$  is closed. In this paper, we prove that, when the space  $T$  is the universal covering of the graph  $\sigma$  consisting in an interval attached by one of its endpoint to a circle (see Figure 1), then the rotation set is the union of finitely many compact intervals. Moreover, all rational points  $r$  in  $\text{Rot}(F)$  are rotation numbers of periodic (mod 1) points. It turns out that the proofs extend to a class of maps on graphs that we call  $\sigma$ -like maps, which are defined in Section 2.3.

## 2 Definitions and elementary properties

### 2.1 Lifted graphs

A (*topological*) *finite graph* is a compact connected set  $G$  containing a finite subset  $V$  such that each connected component of  $G \setminus V$  is homeomorphic to an open interval.

The aim of this section is to define in detail the class of *lifted graphs* where we develop the rotation theory. They are obtained from a topological graph by unwinding one of its loops. This gives a new space that contains a subset homeomorphic to the real line and that is “invariant by a translation” (see Figures 1 and 2). In [2], a larger class of spaces called *lifted spaces* is defined.

**Definition 2.1** Let  $T$  be a connected closed topological space. We say that  $T$  is a *lifted graph* if there exist a homeomorphism  $h$  from  $\mathbb{R}$  into  $T$ , and a homeomorphism  $\tau: T \rightarrow T$  such that

- i)  $\tau(h(x)) = h(x + 1)$  for all  $x \in \mathbb{R}$ ,
- ii) the closure of each connected component of  $T \setminus h(\mathbb{R})$  is a finite graph that intersects  $h(\mathbb{R})$  at a single point,
- iii) the number of connected components  $C$  of  $T \setminus h(\mathbb{R})$  such that  $\overline{C} \cap h([0, 1]) \neq \emptyset$  is finite.

The class of all lifted graphs will be denoted by  $\mathbf{T}^\circ$ .

To simplify the notation, in the rest of the paper we identify  $h(\mathbb{R})$  with  $\mathbb{R}$  itself. In this setting, the map  $\tau$  can be interpreted as a translation by 1. So, for all  $x \in T$  we write  $x + 1$  to denote  $\tau(x)$ . Since  $\tau$  is a homeomorphism, this notation can be extended by denoting  $\tau^m(x)$  by  $x + m$  for all  $m \in \mathbb{Z}$ .

Because of (ii), not all infinite graphs obtained by unwinding a finite graph with a distinguished loop are lifted graphs. The essential property of this class is the existence of a natural retraction from  $T$  to  $\mathbb{R}$ .

**Definition 2.2** Let  $T \in \mathbf{T}^\circ$ . The retraction  $r: T \rightarrow \mathbb{R}$  is the continuous map defined as follows. When  $x \in \mathbb{R}$ , then  $r(x) = x$ . When  $x \notin \mathbb{R}$ , there exists a connected component  $C$  of  $T \setminus \mathbb{R}$  such that  $x \in C$  and  $\overline{C}$  intersects  $\mathbb{R}$  at a single point  $z$ , and we let  $r(x) = z$ .

## 2.2 Maps of degree 1 and rotation numbers

A standard approach to study the periodic points and orbits of a graph map is to work at lifting level with the periodic (mod 1) points. The results on the lifted graph can obviously be pulled back to the original graph (see [2]). Moreover, the rotation numbers have a signification only for maps of degree 1, as in the case of circle maps. In this paper, we deal only with maps of degree 1 on lifted graphs.

**Definition 2.3** Let  $T \in \mathbf{T}^\circ$ . A continuous map  $F: T \rightarrow T$  is of degree 1 if  $F(x+1) = F(x) + 1$  for all  $x \in T$ .

A point  $x \in T$  is called *periodic (mod 1)* for  $F$  if there exists a positive integer  $n$  such that  $F^n(x) \in x + \mathbb{Z}$ . The *period* of  $x$  is the least integer  $n$  satisfying this property, that is,  $F^n(x) \in x + \mathbb{Z}$  and  $F^i(x) \notin x + \mathbb{Z}$  for all  $1 \leq i \leq n-1$ .

The next easy lemma summarises the basic properties of maps of degree 1 (see for instance [1, Section 3.1]).

**Lemma 2.4** Let  $T \in \mathbf{T}^\circ$  and  $F: T \rightarrow T$  a continuous map of degree 1. The following statements hold for  $n \in \mathbb{N}$ ,  $k \in \mathbb{Z}$  and  $x \in T$ :

- i)  $F^n(x+k) = F^n(x) + k$ .
- ii)  $(F+k)^n(x) = F^n(x) + kn$ .
- iii) If  $G: T \rightarrow T$  is another continuous map of degree 1, then  $F \circ G$  is a map of degree 1. In particular,  $F^n$  is of degree 1 for all  $n \geq 1$ .

We define three types of rotation numbers.

**Definition 2.5** Let  $T \in \mathbf{T}^\circ$ ,  $F: T \rightarrow T$  a continuous map of degree 1 and  $x \in T$ . We set

$$\underline{\rho}_F(x) = \liminf_{n \rightarrow +\infty} \frac{r \circ F^n(x) - r(x)}{n} \quad \text{and} \quad \overline{\rho}_F(x) = \limsup_{n \rightarrow +\infty} \frac{r \circ F^n(x) - r(x)}{n}.$$

When  $\underline{\rho}_F(x) = \overline{\rho}_F(x)$  then this number will be denoted by  $\rho_F(x)$  and called the *rotation number of  $x$* .

We now give some elementary properties of rotation numbers (see [2, lemma 1.9]).

**Lemma 2.6** Let  $T \in \mathbf{T}^\circ$ ,  $F: T \rightarrow T$  a continuous map of degree 1,  $x \in T$ ,  $k \in \mathbb{Z}$  and  $n \in \mathbb{N}$ .

- i)  $\overline{\rho}_F(x+k) = \overline{\rho}_F(x)$ .
- ii)  $\overline{\rho}_{(F+k)}(x) = \overline{\rho}_F(x) + k$ .
- iii)  $\overline{\rho}_{F^n}(x) = n\overline{\rho}_F(x)$ .

The same statements hold with  $\underline{\rho}$  instead of  $\overline{\rho}$ .

An important object that synthesises all the information about rotation numbers is the *rotation set* (i.e., the set of *all* rotation numbers). Since we have three types of rotation numbers, we have several kinds of rotation sets.

**Definition 2.7** Let  $T \in \mathbf{T}^\circ$  and  $F: T \rightarrow T$  a continuous map of degree 1. For  $S \subset T$  we define the following *rotation sets*:

$$\begin{aligned}\text{Rot}_S^+(F) &= \{\overline{\rho}_F(x) \mid x \in S\}, \\ \text{Rot}_S^-(F) &= \{\underline{\rho}_F(x) \mid x \in S\}, \text{ and} \\ \text{Rot}_S(F) &= \{\rho_F(x) \mid x \in S \text{ and } \rho_F(x) \text{ exists}\}.\end{aligned}$$

When  $S = T$ , we omit the subscript and we write  $\text{Rot}^+(F)$ ,  $\text{Rot}^-(F)$  and  $\text{Rot}(F)$  instead of  $\text{Rot}_T^+(F)$ ,  $\text{Rot}_T^-(F)$  and  $\text{Rot}_T(F)$ , respectively.

### 2.3 $\sigma$ -like maps

Let  $T \in \mathbf{T}^\circ$  and  $F: T \rightarrow T$  a continuous map of degree 1. Define

$$T_{\mathbb{R}} = \overline{\bigcup_{n \geq 0} F^n(\mathbb{R})}$$

and  $X_F = \overline{T \setminus T_{\mathbb{R}}} \cap r^{-1}([0, 1))$ . Then  $T_{\mathbb{R}} \in \mathbf{T}^\circ$  (Lemma 5.2 in [2]),  $X_F$  is composed of finitely many finite graphs and  $T = T_{\mathbb{R}} \cup (X_F + \mathbb{Z})$ .

If  $T$  is the lifting of the graph  $\sigma$  (see Figure 1), then  $X_F$  is, either empty, or an interval with an endpoint in  $T_{\mathbb{R}}$ . Maps with the same property will be called  *$\sigma$ -like maps*.

**Definition 2.8** Let  $T \in \mathbf{T}^\circ$  and  $F: T \rightarrow T$  a continuous map of degree 1. If  $X_F$  is, either empty, or a nonempty interval such that  $X_F \cap T_{\mathbb{R}}$  is one of the endpoints of  $X_F$ , we say that  $F$  is a  *$\sigma$ -like map* and we write  $F \in \mathcal{C}_1^\sigma(T)$ .

**Remark 2.9** If  $F$  is a  $\sigma$ -like map then so is  $F^n$ , because  $X_{F^n} \subset X_F$ .

This paper is devoted to the study of the rotation set of  $\sigma$ -like maps when  $X_F \neq \emptyset$ . The study of the rotation set  $\text{Rot}_{T_{\mathbb{R}}}(F)$  has already been done in [2].

## 3 Positive covering

Let  $F \in \mathcal{C}_1^\sigma(T)$ . The interval  $X_F$ , when it is not empty, may be endowed with two opposite orders. We choose the one such that  $\min X_F$  is the one-point intersection  $X_F \cap T_{\mathbb{R}}$ . The retraction map  $r_X: T \rightarrow X_F$  can be defined in a natural way by  $r_X(x) = x$  if  $x \in X_F$  and  $r_X(x) = \min X_F$  if  $x \in T_{\mathbb{R}}$ .

The notion of positive covering for subintervals of  $\mathbb{R}$  has been introduced in [2]. It can be extended for subintervals of any subset of  $T$  on which a retraction can be defined. In this paper, we shall use positive covering on  $X_F$ . All properties of positive covering remain valid in this context. In particular, if a compact interval  $I$  positively  $F$ -covers itself, then  $F$  has a fixed point in  $I$  (Proposition 3.5).

**Definition 3.1** Let  $T \in \mathbf{T}^\circ$ ,  $F \in \mathcal{C}_1^\sigma(T)$ ,  $I, J$  two non empty compact subintervals of  $X_F$ ,  $n$  a positive integer and  $p \in \mathbb{Z}$ . We say that  $I$  *positively  $F^n$ -covers*  $J + p$  and we write  $I \xrightarrow[F^n]{+} J + p$  if there exist  $x \leq y$  in  $I$  such that  $r_X(F^n(x) - p) \leq \min J$  and  $\max J \leq r_X(F^n(y) - p)$ . In this situation, we also say that  $I + q$  positively  $F^n$ -covers  $J + p + q$  for all  $q \in \mathbb{Z}$ .

**Remark 3.2** If  $F^n(x) \in T_{\mathbb{R}}$  and  $J \subset X_F$ , then the inequality  $r_X(F^n(x) - p) \leq \min J$  is automatically satisfied. We shall often use this remark to prove that an interval positively covers another.

We introduce some definitions in order to handle sequences of positive coverings.

**Definition 3.3** Let  $T \in \mathbf{T}^\circ$  and  $F \in \mathcal{C}_1^\sigma(T)$ . If we have the following sequence of positive coverings:

$$\mathcal{C}: I_0 + p_0 \xrightarrow{F^{n_1}} I_1 + p_1 \xrightarrow{F^{n_2}} I_2 + p_2 \cdots \cdots I_{k-1} + p_{k-1} \xrightarrow{F^{n_k}} I_k + p_k,$$

(where  $I_0, \dots, I_k$  are non empty compact subintervals of  $X_F$ ,  $n_1, \dots, n_k$  are positive integers and  $p_0, \dots, p_k \in \mathbb{Z}$ ), then  $\mathcal{C}$  is called a *chain* of intervals for  $F$ . Its *length* is  $L_F(\mathcal{C}) = n_1 + \dots + n_k$ , and its *weight* is  $W_F(\mathcal{C}) = p_k - p_0$ . A point  $x$  *follows* the chain  $\mathcal{C}$  if  $F^{n_1+\dots+n_i}(x) \in I_i + p_i$  for all  $0 \leq i \leq k$ .

If  $i \in \mathbb{Z}$ , the chain  $\mathcal{C} + i$  is the translation of  $\mathcal{C}$ , that is

$$\mathcal{C} + i: I_0 + p_0 + i \xrightarrow{F^{n_1}} I_1 + p_1 + i \xrightarrow{F^{n_2}} \cdots \cdots I_{k-1} + p_{k-1} + i \xrightarrow{F^{n_k}} I_k + p_k + i.$$

If  $\mathcal{C}'$  is another chain of intervals beginning with  $I_k + p$  for some  $p \in \mathbb{Z}$ , then  $\mathcal{C}\mathcal{C}'$  is the concatenation of  $\mathcal{C}$  and  $(\mathcal{C}' - p + p_k)$ . If  $I_k = I_0$ , then  $\mathcal{C}^n$  is the  $n$ -times concatenation  $\mathcal{C} \cdots \mathcal{C}$  if  $n \geq 1$  and  $\mathcal{C}^0$  is the empty chain.

The next properties are straightforward.

**Lemma 3.4** Let  $T \in \mathbf{T}^\circ$  and  $F \in \mathcal{C}_1^\sigma(T)$ .

- If  $\mathcal{C}$  is a chain of intervals for  $F^n$ , then it is also a chain of intervals for  $F$  and  $L_F(\mathcal{C}) = nL_{F^n}(\mathcal{C})$  and  $W_F(\mathcal{C}) = W_{F^n}(\mathcal{C})$ . Since the weight is independent of the power of the map, we shall denote it by  $W(\mathcal{C})$ .
- If  $\mathcal{C}, \mathcal{C}'$  are two chains of intervals for  $F$  that can be concatenated, then  $L_F(\mathcal{C}\mathcal{C}') = L_F(\mathcal{C}) + L_F(\mathcal{C}')$  and  $W(\mathcal{C}\mathcal{C}') = W(\mathcal{C}) + W(\mathcal{C}')$ .

The next proposition is [2, Proposition 2.3] (rewritten in some less general form).

**Proposition 3.5** Let  $T \in \mathbf{T}^\circ$ ,  $F \in \mathcal{C}_1^\sigma(T)$  and  $\mathcal{C}$  a chain of subintervals of  $X_F$  such that  $\mathcal{C}$  starts with some interval  $I_0$  and ends with a translation of  $I_0$  (i.e.,  $I_0 + p$  for some  $p \in \mathbb{Z}$ ). Then there exists a point  $x_0$  following the chain  $\mathcal{C}$  such that  $F^{L_F(\mathcal{C})}(x_0) = x_0 + W(\mathcal{C})$ .

The next lemma says that if two intervals  $I, J$  both positively cover translations of  $I$  and  $J$ , then every rational number in the interval corresponding to this “horseshoe” can be obtained as a rotation number of a periodic (mod 1) point. This will be a key tool.

**Lemma 3.6** Let  $T \in \mathbf{T}^\circ$ ,  $G \in \mathcal{C}_1^\sigma(T)$ ,  $I, J$  two non empty compact subintervals of  $X_G$  and  $m_1, m_2 \in \mathbb{Z}$  such that

$$\begin{aligned} I \xrightarrow{G} I + m_1 \quad \text{and} \quad I \xrightarrow{G} J + m_1, \\ J \xrightarrow{G} I + m_2 \quad \text{and} \quad J \xrightarrow{G} J + m_2. \end{aligned}$$

Suppose that  $m_1 \leq m_2$ . For every  $p/q \in [m_1, m_2]$ , there exists  $\mathcal{C}$  a chain of intervals for  $G$  in which all the intervals are translations of  $I$  and  $J$ , and  $p/q = W(\mathcal{C})/L_G(\mathcal{C})$ . Moreover, there exists a periodic (mod 1) point  $x \in I \cup J$  such that  $\rho_G(x) = p/q$ .

If  $p/q \neq m_2$ , then  $\mathcal{C}$  can be chosen such that the first interval is  $I$  and the last interval is a translation of  $I$ , and the periodic (mod 1) point  $x$  can be chosen in  $I$ .

*Proof.* By considering  $G - m_1$  instead of  $G$ , we may suppose that  $m_1 = 0$  (use Lemma 2.6). Since  $p/q \in [0, m_2]$ , we have  $0 \leq p \leq m_2 q$ . If  $p/q = m_2$ , we take  $\mathcal{C}: J \xrightarrow{+}_G J + m_2$ . By Proposition 3.5, there exists a point  $x \in J$  such that  $F(x) = x + m_2$ , and hence  $\rho_G(x) = m_2$ .

If  $p = 0$ , we take  $\mathcal{C}: I \xrightarrow{+}_G I$ . If  $1 \leq p \leq m_2 q - 1$ , we take

$$\mathcal{C}: (I \xrightarrow{+}_G I)^{m_2 q - 1 - p} (I \xrightarrow{+}_G J) (J \xrightarrow{+}_G J + m_2)^{p-1} (J \xrightarrow{+}_G I + m_2)$$

In these two cases, it is straightforward that  $W(\mathcal{C})/L_G(\mathcal{C}) = p/q$ . By Proposition 3.5, there exists a point  $x \in I$  such that  $G^{L_G(\mathcal{C})}(x) = x + W(\mathcal{C})$ , and so  $x$  is periodic (mod 1) and  $\rho_G(x) = W(\mathcal{C})/L_G(\mathcal{C}) = p/q$ .  $\square$

## 4 Study of the rotation set of $F$

Let  $T \in \mathbf{T}^\circ$  and  $F \in \mathcal{C}_1^\sigma(T)$ . Since  $T = T_{\mathbb{R}} \cup (X_F + \mathbb{Z})$ , it is clear that  $\text{Rot}(F) = \text{Rot}_{T_{\mathbb{R}}}(F) \cup \text{Rot}_{X_F}(F)$ , and the same holds with  $\text{Rot}^+$  and  $\text{Rot}^-$ . The rotation set  $\text{Rot}_{T_{\mathbb{R}}}(F)$  has been studied in [2]. Consequently, it remains to study the rotation set  $\text{Rot}_{X_F}(F)$ . The next theorem summarises the properties of  $\text{Rot}_{\mathbb{R}}(F)$  (see Theorems 3.1, 3.11, 5.7 and 5.18 in [2]).

**Theorem 4.1** *Let  $T \in \mathbf{T}^\circ$  and  $F: T \rightarrow T$  a continuous map of degree 1. Then  $\text{Rot}_{\mathbb{R}}(F)$  is a non empty compact interval and, if  $T_{\mathbb{R}}$  is defined as above,  $\text{Rot}_{T_{\mathbb{R}}}(F) = \text{Rot}_{T_{\mathbb{R}}}^+(F) = \text{Rot}_{T_{\mathbb{R}}}^-(F) = \text{Rot}_{\mathbb{R}}(F)$ . Moreover, if  $r \in \text{Rot}_{\mathbb{R}}(F) \cap \mathbb{Q}$ , then there exists a periodic (mod 1) point  $x \in T_{\mathbb{R}}$  such that  $\rho_F(x) = r$ .*

### 4.1 Partition of $X_F$

If  $F^n(x) \in T_{\mathbb{R}}$  for some  $n$ , then  $\rho_F(x) \in \text{Rot}_{T_{\mathbb{R}}}(F)$ . We already know the properties of  $\text{Rot}_{T_{\mathbb{R}}}(F)$ . Therefore, it is sufficient to consider the points  $x \in X_F$  whose orbit does not fall in  $T_{\mathbb{R}}$ , or equivalently the points in  $X_\infty = \{x \in X_F \mid \forall n \geq 1, F^n(x) \in X_F + \mathbb{Z}\}$ .

Our first step consists in dividing  $X_F$  according to the translations of the images with respect to  $X_F + \mathbb{Z}$ . If  $F(x) \in X_F + p$  and  $F(y) \in X_F + p'$  with  $p \neq p'$ , then necessarily there is a gap between  $x$  and  $y$  by continuity. Thus we can include the points  $\{x \in X_F \mid F(x) \in X_F + \mathbb{Z}\}$  in a finite union of disjoint compact intervals such that, for each  $I$  among these intervals, there is a unique integer  $p$  satisfying  $F(I) \cap (X_F + p) \neq \emptyset$ .

**Lemma 4.2** *Let  $T \in \mathbf{T}^\circ$  and  $F \in \mathcal{C}_1^\sigma(T)$ . There exist an integer  $N \geq 0$ , non empty compact subintervals  $X_1, \dots, X_N$  of  $X_F$  and integers  $p_1, \dots, p_N$  in  $\mathbb{Z}$  such that*

- i)  $X_1 < X_2 < \dots < X_N$ ,
- ii)  $F(X_i) \subset (X_F + p_i) \cup T_{\mathbb{R}}$  for all  $1 \leq i \leq N$ ,
- iii)  $F(\min X_i) = \min X_F + p_i$  for all  $1 \leq i \leq N$ ,
- iv)  $p_{i+1} \neq p_i$  for all  $1 \leq i \leq N - 1$ ,
- v)  $F(X_F \setminus (X_1 \cup \dots \cup X_N)) \cap (X_F + \mathbb{Z}) = \emptyset$ .

*Proof.* If  $F(X_F) \cap (X_F + \mathbb{Z}) = \emptyset$ , we take  $N = 0$  and there is nothing to do. Otherwise, we can define  $a_1 = \min\{x \in X_F \mid F(x) \in X_F + \mathbb{Z}\}$  and  $p_1 \in \mathbb{Z}$  such that  $F(a_1) \in X_F + p_1$ . We define

$$b_1 = \max\{x \in [a_1, \max X_F] \mid F(x) \in X_F + p_1 \text{ and } F([a_1, x]) \subset (X_F + p_1) \cup T_{\mathbb{R}}\},$$

and  $X_1 = [a_1, b_1]$ . Then  $X_1$  satisfies (ii). Moreover,  $F(\min X_F) \in T_{\mathbb{R}}$  because  $\min X_F \in T_{\mathbb{R}}$ , which implies that  $F([\min X_F, a_1])$  contains  $\min X_F + p_1$ . Thus  $F(a_1) = \min X_F + p_1$  by minimality of  $a_1$ , which is (iii) for  $X_1$ .

We define  $X_2, \dots, X_M$  inductively. Suppose that  $X_i = [a_i, b_i]$  and  $p_i$  are already defined and that  $b_i$  verifies:

$$b_i = \max\{x \in [a_i, \max X_F] \mid F(x) \in X_F + p_i \text{ and } F([a_i, x]) \subset (X_F + p_i) \cup T_{\mathbb{R}}\}.$$

If  $F((b_i, \max X_F]) \cap (X_F + \mathbb{Z}) = \emptyset$ , then we take  $N = i$  and the construction is over. Otherwise, we define

$$a_{i+1} = \inf\{x \in (b_i, \max X_F] \mid F(x) \in X_F + \mathbb{Z}\}. \quad (1)$$

We first show that  $a_{i+1}$  is actually defined by a minimum in (1). By definition, there exists a sequence of points  $x_n \in (b_i, \max X_F]$  tending to  $a_{i+1}$  and such that  $F(x_n) \in X_F + \mathbb{Z}$ . Let  $m_n \in \mathbb{Z}$  such that  $F(x_n) \in X_F + m_n$ . By continuity,  $\lim_{n \rightarrow +\infty} r \circ F(x_n) = r \circ F(a_{i+1})$ . Since  $r \circ F(x_n) = r(\min X_F) + m_n$ , this implies that the sequence of integers  $(m_n)_{n \geq 0}$  is ultimately constant, and equal to some integer  $p_{i+1}$ . Then  $F(a_{i+1}) = \lim_{n \rightarrow +\infty} F(x_n) \in X_F + p_{i+1}$ . By continuity,  $F([a_{i+1}, x_n]) \subset (X_F + p_{i+1}) \cup T_{\mathbb{R}}$  for all  $n$  large enough. Moreover,  $F((b_i, a_{i+1})) \cap (X_F + \mathbb{Z}) = \emptyset$  by definition of  $a_{i+1}$ . If  $p_{i+1} = p_i$ , then, for  $n$  large enough, we would have

$$F(x_n) \in X_F + p_i \text{ and } F([b_i, x_n]) \subset (X_F + p_i) \cup T_{\mathbb{R}},$$

which would contradict the definition of  $b_i$  because  $x_n > b_i$ . Hence  $p_{i+1} \neq p_i$ . This implies that  $a_{i+1} > b_i$ . Since  $F((b_i, a_{i+1}))$  is non empty and included in  $T_{\mathbb{R}}$ , necessarily  $F(a_{i+1})$  is equal to  $\min X_F + p_{i+1}$  by minimality of  $a_{i+1}$ .

Finally, we define

$$b_{i+1} = \max\{x \in [a_{i+1}, \max X_F] \mid F(x) \in X_F + p_{i+1} \text{ and } F([a_{i+1}, x]) \subset (X_F + p_{i+1}) \cup T_{\mathbb{R}}\},$$

and  $X_{i+1} = [a_{i+1}, b_{i+1}]$ . Then  $X_{i+1} > X_i$  and (ii), (iii) and (iv) are satisfied.

By uniform continuity of  $r \circ F$  on the compact set  $X_F$ , there exists  $\delta > 0$  such that, if  $x, y \in X_F$  with  $|x - y| < \delta$ , then  $|r \circ F(x) - r \circ F(y)| < 1$ . This implies that  $|a_{i+1} - b_i| \geq \delta$ , which ensures that the number of intervals  $X_i$  is finite, and the construction ultimately stops. By construction, (v) is satisfied.  $\square$

**Remark 4.3** i) The fact that the sets  $X_1, \dots, X_N$  are intervals is very important because it will allow us to use positive coverings. Note that we cannot ask that  $F(X_i) \subset (X_F + p_i)$ , even if we do not require that  $p_{i+1} \neq p_i$ . Indeed, if  $\min X_F$  is a fixed point, the map  $F$  may oscillate infinitely many times between  $X_F$  and  $T_{\mathbb{R}}$  in any neighbourhood of  $\min X_F$ , and in this case the number of connected components of  $F(X_F) \cap X_F$  is infinite.

ii) In the partition of  $X_F$  into  $X_1, \dots, X_N, X_F \setminus (X_1 \cup \dots \cup X_N)$ , the set  $X_F \setminus (X_1 \cup \dots \cup X_N)$  plays the role of “dustbin”, and we can code the itinerary of every point in  $X_{\infty}$  with respect to  $X_1, \dots, X_N$ . More precisely, if  $F^n(x) \in X_F \setminus (X_1 \cup \dots \cup X_N) + \mathbb{Z}$ , then  $x \notin X_{\infty}$ . Therefore, for every  $x \in X_{\infty}$ ,  $\forall n \geq 0, \exists ! \omega_n \in \{1, \dots, N\}$  such that  $F^n(x) \in X_{\omega_n} + \mathbb{Z}$ . The rotation number of  $x$  can be deduced from this coding sequence because  $\forall n \geq 0, F^n(x) \in X_{\omega_n} + p_{\omega_0} + \dots + p_{\omega_{n-1}}$ .

iii) It can additionally be shown that  $F(\max X_i) = \min X_F + p_i$  for all  $1 \leq i \leq N - 1$  and, for  $i = N$ , either  $F(\max X_N) = \min X_F + p_N$ , or  $\max X_N = \max X_F$ .

## 4.2 Periodic (mod 1) points associated to the endpoints of rotation sets

When proving that every rational number in the rotation set is the rotation number of a periodic (mod 1) point, we shall make a distinction between rational numbers in the interior and rational numbers in the boundary (the same distinction is necessary to deal with  $\text{Rot}_{\mathbb{R}}(F)$  [2]). For rational numbers in the boundary, harder to handle, we shall need the following lemma, which is the analogous of [2, Lemma 5.15] in our context. This lemma is aimed to be applied first with  $T' = T_{\mathbb{R}}$ ,  $Y = X_F$  and  $Z = X_1$ , where  $X_1$  is defined in Lemma 4.2. After dealing with  $X_1$ , an induction will be done to deal with  $X_2, \dots, X_N$ , that is why the lemma is stated with general notations.

**Lemma 4.4** *Let  $T \in \mathbf{T}^\circ$  and  $F \in \mathcal{C}_1^\sigma(T)$ . Let  $T'$  be a closed connected subset of  $T$  such that  $T_{\mathbb{R}} \subset T'$ ,  $T' + 1 = T'$  and  $F(T') \subset T'$ . Let  $Y$  denote the compact subinterval of  $X_F$  equal to  $\overline{T \setminus T'} \cap r^{-1}([0, 1))$  and define  $Y_\infty = \{x \in Y \mid \forall n \geq 1, F^n(x) \in Y + \mathbb{Z}\}$ . Let  $Z$  be a compact subinterval of  $Y$  such that  $F(\min Z) = \min Y$  and  $F(Z) \cap Z \neq \emptyset$ . Assume that  $\inf \text{Rot}_{Z \cap Y_\infty}(F) \geq 0$  and*

$$\forall x \in \overline{\bigcup_{n \geq 0} (F^n(Z) + \mathbb{Z})} \cap Y_\infty, \forall n \geq 1, F^n(x) \neq x.$$

*Then  $\inf \text{Rot}_{Z \cap Y_\infty}(F) > 0$ .*

*Proof.* We note  $x_0 = \min Y$ . By assumption,  $F^n(x_0) \in T'$  for all  $n \geq 0$ . Let

$$Y' = \overline{\bigcup_{n \geq 0} (F^n(Z) + \mathbb{Z})} \cap Y.$$

For all  $n \geq 1$ ,  $F^n(x_0) \in T'$ , which implies that  $(F^n(Z) + \mathbb{Z}) \cap Y$  is an interval containing  $x_0$ . In addition,  $F(Z) \cap Z \neq \emptyset$  by assumption. This implies that  $Y'$  is a compact subinterval of  $Y$  containing  $x_0$ .

Let  $x \in Y'$  and  $k \in \mathbb{Z}$  such that  $F(x) \in Y + k$ . Suppose that  $F(x) - k \geq x$ . Then  $[x_0, x] \xrightarrow{+}_F [x_0, x] + k$  and, according to Proposition 3.5, there exists  $y \in [x_0, x] \subset Y'$  such that  $F(y) = y + k$ . Hence  $y \in Y_\infty$  and  $\rho_F(y) = k$ . By assumption, this is possible only if  $k > 0$ . Hence

$$\text{if } x \in Y' \text{ and } F(x) \in Y + k \text{ with } k \leq 0, \text{ then } F(x) - k < x. \quad (2)$$

We are going to prove that there exists an integer  $N_0$  such that

$$\text{if } y \in Y' \text{ verifies } \forall 1 \leq n \leq N, \exists k_n \leq 0, F^n(y) \in Y + k_n, \text{ then } N \leq N_0. \quad (3)$$

Let  $y$  verify the assumption of (3) and note  $k_0 = 0$ . By Equation (2),  $(F^i(y) - k_i)_{0 \leq i \leq N}$  is a decreasing sequence in  $Y$ . Since  $Y'$  is an interval containing  $y$  and  $\min Y$ , all the points of this sequence belong to  $Y'$  too.

According to (2), for all  $k \leq 0$  and all  $x \in Y'$ ,  $F(x) - k \neq x$ . If  $d$  denotes a distance on  $T$ , then

$$\forall k \leq 0, \delta_k = \inf\{d(x, F(x) - k) \mid x \in Y'\} > 0$$

because  $Y'$  is compact. Moreover, the set of integers  $k$  such that  $F(Y') \cap (Y + k) \neq \emptyset$  is finite, and so

$$\delta = \inf_{k \leq 0} \delta_k > 0.$$



Consequently,  $(F^i(y) - k_i)_{0 \leq i \leq N}$  is a decreasing sequence in  $Y'$  and, for all  $0 \leq i \leq N - 1$ ,  $d(F^{i+1}(y) - k_{i+1}, F^i(y) - k_i) \geq \delta$ . This implies that  $\text{diam}(Y') \geq N\delta$ . This proves that (3) holds if  $N_0 \geq \frac{\text{diam}(Y')}{\delta}$ .

Now, let  $x \in Z \cap Y_\infty$ . For all  $n \geq 0$ ,  $F^n(x) \in Y' + \mathbb{Z}$ . According to (3), there exists an increasing sequence of positive integers  $(n_i)_{i \geq 1}$  and integers  $(k_i)_{i \geq 1}$  such that

$$\forall i \geq 1, F^{n_i}(x) \in Y' + k_i, n_{i+1} - n_i \leq N_0 \quad \text{and} \quad k_{i+1} \geq k_i + 1.$$

This implies that  $\bar{\rho}_F(x) \geq 1/N_0 > 0$ . This concludes the proof of the lemma.  $\square$

Let us restate Lemma 4.4 when  $T' = T_{\mathbb{R}}$ ,  $Y = X_F$  and  $Z = X_1$ : if  $\inf \text{Rot}_{X_1 \cap X_\infty}(F) = 0$  and  $F(X_1) \cap X_1 \neq \emptyset$ , then there exist a point  $x \in \bigcup_{n \geq 0} (F^n(X_1) + \mathbb{Z}) \cap X_\infty$  and an integer  $n \geq 1$  such that  $F^n(x) = x$ . Note that when  $p_1 = 0$ , the assumption  $F(X_1) \cap X_1 \neq \emptyset$  is fulfilled as soon as  $\text{Rot}_{X_1 \cap X_\infty}(F) \neq \emptyset$ . Indeed, if  $F(X_1) \cap X_1 = \emptyset$ , then  $F(X_1) \cap (X_F + \mathbb{Z}) \subset [\min X_F, \min X_1)$ , and thus  $X_1 \cap X_\infty = \emptyset$ . More generally, this result implies that, if  $\inf \text{Rot}_{X_1 \cap X_\infty}(F) = p/q$ , then there exists a periodic (mod 1) point  $x \in X_F$  such that  $\rho_F(x) = p/q$  (just apply what precedes to the map  $F^q - p$ ). This is the analogous in our context of [2, Theorem 5.18].

### 4.3 Rotation set of $X_1$

In the sequel, we shall heavily use the fact that  $X_F$  is an interval with an endpoint in  $T_{\mathbb{R}}$ . By definition,  $\min X_F$  belongs to  $T_{\mathbb{R}}$  and  $T_{\mathbb{R}}$  is invariant by  $F$ . Hence  $F(\min X_F) \in T_{\mathbb{R}}$ . Therefore, if  $I$  is a subinterval of  $X_F$  such that  $\min I = \min X_F$  and  $F(I) \cap X_F \neq \emptyset$ , then necessarily  $F(I) \cap X_F$  is an interval containing  $\min X_F$ . This simple observation allows us to study the rotation set of the interval  $X_1$  defined in Lemma 4.2. This is done in Proposition 4.6, by considering  $T' = T_{\mathbb{R}}$  and  $Y_1, \dots, Y_M = X_1, \dots, X_N$ . When this is done for  $X_1$ , the idea is to proceed by induction for the rotation sets of  $X_2, \dots, X_N$ , this is why the proposition is stated with more general notations.

In the proof of Proposition 4.6, we shall need the next, technical lemma.

**Lemma 4.5** *Let  $(n_k)_{k \geq 0}$  be a sequence of real numbers bounded from above by some constant  $C$ . Let  $\varepsilon > 0$ ,*

$$L = \limsup_{k \rightarrow +\infty} \frac{n_0 + \dots + n_{k-1}}{k}$$

*and  $l < L - \varepsilon$ . Then there exists an integer  $k \geq 1$  such that  $\frac{n_0 + \dots + n_{k-1}}{k} \geq L - \varepsilon$  and  $n_k \neq l$ .*

*Proof.* Let  $K$  be an integer such that  $C/K < \varepsilon/2$ . Let

$$E = \left\{ k \in \mathbb{Z}^+ \mid \frac{n_0 + \dots + n_k}{k+1} \geq L - \varepsilon/2 \right\}.$$

The set  $E$  is infinite by definition of  $L$ . We are going to do a proof by absurd. We assume that

$$n_k = l \quad \text{for all } k \in E \text{ such that } k > K. \tag{4}$$

If  $E$  contains all integers  $n \geq N$  for some  $N$ , then  $L = l$ , which is absurd. Thus there exists an integer  $k \geq K$  such that  $k \in E$  and  $k - 1 \notin E$ . We have

$$\frac{n_0 + \dots + n_k}{k+1} = \frac{k}{k+1} \cdot \frac{n_0 + \dots + n_{k-1}}{k} + \frac{1}{k+1} n_k.$$

By definition of  $E$ ,  $\frac{n_0 + \dots + n_k}{k+1} \geq L - \varepsilon/2$  and  $\frac{n_0 + \dots + n_{k-1}}{k} < L - \varepsilon/2$ . Moreover,  $n_k = l < L - \varepsilon/2$  by (4). Thus

$$\frac{n_0 + \dots + n_k}{k+1} < \frac{k}{k+1}(L - \varepsilon/2) + \frac{1}{k+1}(L - \varepsilon/2) = L - \varepsilon/2,$$

which is a contradiction. Therefore, (4) does not hold, and there exists  $k \in E$  such that  $k > K$  and  $n_k \neq l$ . Moreover,

$$\begin{aligned} \frac{n_0 + \dots + n_{k-1}}{k} &= \frac{k+1}{k} \cdot \frac{n_0 + \dots + n_k}{k+1} - \frac{n_k}{k} \\ &> \frac{n_0 + \dots + n_k}{k+1} - \frac{C}{K} \\ &> L - \varepsilon/2 - \varepsilon/2 = L - \varepsilon \end{aligned}$$

Such an integer  $k$  is suitable.  $\square$

**Proposition 4.6** *Let  $T \in \mathbf{T}^\circ$  and  $F \in \mathcal{C}_1^\sigma(T)$ . Let  $T'$  be a connected subset of  $T$  such that  $T_{\mathbb{R}} \subset T'$ ,  $T' + 1 = T'$  and  $F(T') \subset T'$ . Let  $Y$  denote the compact subinterval of  $X_F$  equal to  $T \setminus T' \cap r^{-1}([0, 1))$  and define  $Y_\infty = \{x \in Y \mid \forall n \geq 1, F^n(x) \in Y + \mathbb{Z}\}$ . Let  $Y_1, \dots, Y_M$  be compact subintervals of  $Y$  and  $q_1, \dots, q_M \in \mathbb{Z}$  such that:*

- a)  $Y_1 < \dots < Y_M$ ,
- b)  $F(Y_i) \subset (Y + q_i) \cup T'$  for all  $1 \leq i \leq M$ ,
- c)  $F(\min Y_1) = \min Y \pmod{1}$ ,
- d)  $F(Y \setminus (Y_1 \cup \dots \cup Y_M)) \cap (Y + \mathbb{Z}) = \emptyset$ .

Assume that  $Y_1 \cap Y_\infty \neq \emptyset$ . Then there exists a compact interval  $I \subset \mathbb{R}$  such that:

- i)  $\text{Rot}_{Y_1 \cap Y_\infty}(F) = \text{Rot}_{Y_1 \cap Y_\infty}^+(F) = \text{Rot}_{Y_1 \cap Y_\infty}^-(F) = I$ ,
- ii)  $I$  contains  $q_1$ ,
- iii) there exists  $a \in Y_1 \cap Y_\infty$  such that  $F(a) = a + q_1$  and  $[\min Y_1, a) \cap Y_\infty = \emptyset$ ,
- iv) if  $r \in \text{Int}(I) \cap \mathbb{Q}$  then there exists a periodic  $(\text{mod } 1)$  point  $x \in Y_1 \cap Y_\infty$  with  $\rho_F(x) = r$ .
- v) if  $r \in \partial I \cap \mathbb{Q}$  then there exists a periodic  $(\text{mod } 1)$  point  $x \in \overline{\bigcup_{n \geq 0} (F^n(Y_1) + \mathbb{Z})} \cap Y_\infty$  with  $\rho_F(x) = r$ .

*Proof.* We first prove (iii) under an additional assumption:

$$\text{If } \exists y \in Y_1 \text{ such that } \forall n \geq 1, F^n(y) \in Y_1 + \mathbb{Z}, \text{ then (iii) holds.} \quad (5)$$

Let  $G = F - q_1$  and  $a_0 = \min Y_1$ . Then  $G(Y_1) \subset Y \cup T'$ . For all  $n \geq 0$ ,  $G^n(y) \in Y_1$ , and in particular  $G^n(y) \geq a_0$ . We define inductively a sequence of points  $(a_i)_{i \geq 1}$  such that  $a_i \in [a_{i-1}, y]$ ,  $G^i(a_i) = a_0$  and  $[a_0, a_i) \cap Y_\infty = \emptyset$  for all  $i \geq 1$ .

- Since  $G(a_0) = \min Y$  (by assumption (c)) and  $G(y) \geq a_0$ , we have  $a_0 \in G([a_0, y])$  by continuity. Thus there exists  $a_1 \in [a_0, y]$  such that  $G(a_1) = a_0$ . We choose  $a_1$  minimum with this property, which implies that  $G([a_0, a_1)) \cap Y_1 = \emptyset$ . Hence  $[a_0, a_1) \cap Y_\infty = \emptyset$ .

- Assume that  $a_0, \dots, a_i$  are already defined. Since  $G^{i+1}(a_i) = G(a_0) = \min Y$  and  $G^{i+1}(y) \geq a_0$ , the point  $a_0$  belongs to  $G^{i+1}([a_i, y])$  by continuity. Thus there exists  $a_{i+1} \in [a_i, y]$  such that  $G^{i+1}(a_{i+1}) = a_0$ . We choose  $a_{i+1}$  minimum with this property, which implies that  $G^{i+1}([a_i, a_{i+1})) \cap Y_1 = \emptyset$ . Hence  $[a_i, a_{i+1}) \cap Y_\infty = \emptyset$ . This concludes the construction of  $a_{i+1}$ .

The sequence  $(a_i)_{i \geq 0}$  is non decreasing and contained in the compact interval  $Y_1$ . Therefore  $a = \lim_{i \rightarrow +\infty} a_i$  exists and belongs to  $Y_1$ . Since  $G(a_{i+1}) = a_i$ , we get that  $G(a) = a$ . In other words,  $F(a) = a + q_1$ . This implies that  $a \in Y_\infty$ . Moreover,  $[a_0, a) = \bigcup_{i \geq 0} [a_i, a_{i+1})$ , and thus  $[a_0, a) \cap Y_\infty = \emptyset$ . This proves (5).

We split the rest of the proof into two cases.

**Case 1:**  $F(Y_1) \cap (Y_i + \mathbb{Z}) = \emptyset$  for all  $i \geq 2$  (this includes the case  $M = 1$ ).

Then  $F(Y_1 \cap Y_\infty) \subset Y_1 + q_1$  and  $F^n(Y_1 \cap Y_\infty) \subset Y_1 + nq_1$ . Thus, for all  $x \in Y_1 \cap Y_\infty$ , the rotation number  $\rho_F(x)$  exists and is equal to  $q_1$ . We take  $I = \{q_1\}$  and we get (i) and (ii).

Since  $Y_1 \cap Y_\infty$  is not empty, there exists a point  $y$  such that  $\forall n \geq 0$ ,  $F^n(y) \in Y_1 + \mathbb{Z}$ . Then (5) gives (iii), which implies (v) in the present case, and (iv) is empty.

**Case 2:** there exists  $i \geq 2$  such that  $F(Y_1) \cap (Y_i + \mathbb{Z}) \neq \emptyset$ .

Since  $Y_1 < \dots < Y_M$  and  $F(Y_1) \subset (Y + q_1) \cup T'$ , there exists  $x \in Y_1$  such that  $F(x) \in Y + q_1$  and  $F(x) \geq \max(Y_1 + q_1)$ . Moreover,  $F(\min Y_1) = \min Y + q_1$  by assumption. This implies that

$$Y_1 \xrightarrow[F]{+} Y_1 + q_1. \quad (6)$$

By Proposition 3.5, there exists  $y \in Y_1$  such that  $F(y) = y + q_1$ . Thus we can use (5) to get (iii).

Let  $x \in Y_\infty$ . By assumption d), for all  $n \geq 0$ , there exists  $\omega_n \in \{1, \dots, M\}$  such that  $F^n(x) \in Y_{\omega_n} + \mathbb{Z}$ . The sequence  $(\omega_n)_{n \geq 0}$  is called the *itinerary* of  $x$ . The next two results are straightforward.

$$\forall n \geq 0, F^n(x) \in Y_{\omega_n} + q_{\omega_0} + q_{\omega_1} + \dots + q_{\omega_{n-1}}. \quad (7)$$

$$\bar{\rho}_F(x) = \limsup_{n \rightarrow +\infty} \frac{q_{\omega_0} + q_{\omega_1} + \dots + q_{\omega_{n-1}}}{n} \quad \text{and} \quad \underline{\rho}_F(x) = \liminf_{n \rightarrow +\infty} \frac{q_{\omega_0} + q_{\omega_1} + \dots + q_{\omega_{n-1}}}{n}; \quad (8)$$

if the limit exists, it is  $\rho_F(x)$ .

Let  $S = \sup \text{Rot}_{Y_1 \cap Y_\infty}^+(F)$ . Necessarily,  $S \geq q_1$  because  $q_1 \in \text{Rot}_{Y_1 \cap Y_\infty}(F)$  by (iii). We are going to show that  $[q_1, S] \subset \text{Rot}_{Y_1 \cap Y_\infty}(F)$ . If  $S = q_1$  there is nothing to prove, and so we suppose that  $S > q_1$ . Let  $k$  be an integer such that  $S > q_1 + 1/k$ . Let  $y_k$  be a point in  $Y_1 \cap Y_\infty$  such that  $\bar{\rho}_F(y_k) \geq S - 1/2k$ , and let  $(\omega_n)_{n \geq 0}$  denote the itinerary of  $y_k$ . By (8),  $\limsup_{n \rightarrow +\infty} \frac{q_{\omega_0} + q_{\omega_1} + \dots + q_{\omega_{n-1}}}{n} = \bar{\rho}_F(y_k)$ . Applying Lemma 4.5 with  $L = \bar{\rho}_F(y_k)$ ,  $l = q_1$  and  $\varepsilon = 1/2k$ , we get that there exists an integer  $n$  such that

$$\frac{q_{\omega_0} + q_{\omega_1} + \dots + q_{\omega_{n-1}}}{n} \geq \bar{\rho}_F(y_k) - 1/2k \geq S - 1/k > q_1 \quad \text{and} \quad \omega_n \neq 1. \quad (9)$$

By (7),  $F^n(y_k) \in Y_{\omega_n} + q_{\omega_0} + q_{\omega_1} + \dots + q_{\omega_{n-1}}$ . Since  $\omega_n \neq 1$ , we have  $Y_{\omega_n} > Y_1$ , and thus  $F^n(y_k) - (q_{\omega_0} + q_{\omega_1} + \dots + q_{\omega_{n-1}}) > \max Y_1$ . Moreover,  $F^n(\min Y_1) \in F^{n-1}(\overline{T'}) \subset \overline{T'}$  by assumption c) and invariance of  $T'$ . If we let  $I_k = [\min Y_1, y_k]$ ,  $N_k = q_{\omega_0} + \dots + q_{\omega_{n-1}}$  and  $n_k = n$ , we have then:

$$I_k \xrightarrow[F^{n_k}]{+} Y_1 + N_k. \quad (10)$$

Since  $I_k \subset Y_1$ , Equations (6) and (10) give:

$$\begin{aligned} I_k &\xrightarrow{F^{n_k}} I_k + N_k & \text{and} & & I_k &\xrightarrow{F^{n_k}} Y_1 + N_k, \\ Y_1 &\xrightarrow{F^{n_k}} I_k + n_k q_1 & \text{and} & & Y_1 &\xrightarrow{F^{n_k}} Y_1 + n_k q_1. \end{aligned} \quad (11)$$

Let  $r \in [q_1, S) \cap \mathbb{Q}$ . By (9), there exists an integer  $k$  such that  $\frac{N_k}{n_k} > r$ . We apply Lemma 3.6 with  $I = Y_1$ ,  $J = I_k$ ,  $G = F^{n_k}$ ,  $m_1 = n_k q_1$ ,  $m_2 = N_k$  and  $p/q = r n_k \in [q_1 n_k, N_k)$ :

$$\begin{aligned} \exists \mathcal{C}_r &\text{ a chain of intervals for } F \text{ whose first and last intervals are translations of } Y_1, \\ &\text{such that } r = W(\mathcal{C}_r)/L_F(\mathcal{C}_r), \end{aligned} \quad (12)$$

and

$$\text{there exists a periodic (mod 1) point } x \in Y_1 \text{ with } \rho_F(x) = \frac{1}{n_k} \rho_{F^{n_k}}(x) = r. \quad (13)$$

We need to show that  $x \in Y_\infty$ . This is a consequence of the following fact.

**Fact.** Let  $x \in Y_1$  such that  $F^n(x) \in Y_1 + \mathbb{Z}$  for infinitely many  $n$ . Then, either  $x \in Y_\infty$ , or there exists  $n$  such that  $F^n(x)$  is a fixed (mod 1) point in  $Y_1 \cap Y_\infty \cap T'$ .

*Proof of the fact.* If  $\forall n \geq 0$ ,  $F^n(x) \notin T'$ , then  $x \in Y_\infty$  by assumption d) of the proposition. Suppose on the contrary that there exists  $n_0$  such that  $F^{n_0}(x) \in T'$ . Hence  $F^n(x) \in T'$  for all  $n \geq n_0$ . Let  $e = \min Y$ . By definition of  $Y$ , the set  $Y \cap T'$  is included in  $\{e\}$  (we have not supposed that  $T'$  is closed, and thus  $Y \cap T'$  may be empty). Note that  $Y_1 \cap T'$  is empty if  $\min Y_1 > e$ . By assumption, there exists  $n_1 \geq n_0$  such that  $F^{n_1}(x) \in Y_1 + \mathbb{Z}$ . Hence  $F^{n_1}(x) \in (Y_1 + \mathbb{Z}) \cap T'$ . This implies that  $F^{n_1}(x)$  is equal to  $e \pmod{1}$ ,  $\min Y_1 = e \in T'$ , and  $e \in T'$ . By assumption c) of the proposition,  $F(\min Y_1) = e \pmod{1}$ . Thus,  $e$  is a fixed (mod 1) point in  $Y_1$ , and so  $e \in Y_\infty$ . This ends the proof of the fact.  $\square$

Now, let  $\alpha \in [q_1, S]$ . To show that there exists  $x \in Y_1 \cap Y_\infty$  with  $\rho_F(x) = \alpha$ , we use the same method as in the proof of [2, Theorem 3.7]. We choose a sequence of rational numbers  $r_i$  in  $[q_1, S) \cap \mathbb{Q}$  such that  $\lim_{i \rightarrow +\infty} r_i = \alpha$  for all  $i \geq 1$ . Let  $\mathcal{C}_{r_i}$  be the chain of intervals given by (4.3). We define

$$\mathcal{D}_n = (\mathcal{C}_{r_1})^{i_1} (\mathcal{C}_{r_2})^{i_2} \dots (\mathcal{C}_{r_n})^{i_n}.$$

Let  $A_n$  be the set of points that follow the chain  $\mathcal{D}_n$ . This set is compact by definition, and it is not empty because it contains at least a periodic (mod 1) point by Proposition 3.5. Moreover,  $A_{n+1} \subset A_n$ . Therefore  $A = \bigcap_{n \geq 1} A_n \neq \emptyset$ . In the proof of [2, Theorem 3.7], it is shown that, if the sequence  $(i_n)_{n \geq 1}$  increases sufficiently fast and  $(|r_{i_n} - \alpha|)_{n \geq 1}$  is non decreasing then, for all  $x \in A$ ,  $\rho_F(x) = \alpha$ .

Moreover, the fact above implies that, for every  $x \in A$  there exists  $n$  such that  $F^n(x) \in Y_1 \cap Y_\infty$ . Obviously,  $\rho_F(x) = \rho_F(F^n(x))$ . This proves that  $[q_1, \sup \text{Rot}_{Y_1 \cap Y_\infty}^+(F)]$  is included in  $\text{Rot}_{Y_1 \cap Y_\infty}(F)$ ; in addition, (iv) holds for all rational numbers  $r \in [q_1, \sup \text{Rot}_{Y_1 \cap Y_\infty}^+(F))$  by (13). We can apply the same method to  $[\inf \text{Rot}^- Y_1 \cap Y_\infty(F), q_1]$ . Finally, if we define  $I = [\inf \text{Rot}_{Y_1 \cap Y_\infty}^-(F), \sup \text{Rot}_{Y_1 \cap Y_\infty}^+(F)]$ , we get that  $I \subset \text{Rot}_{Y_1 \cap Y_\infty}(F)$ ,  $q_1 \in I$  (which is (ii)) and (iv) holds for all  $r \in \text{Int}(I) \cap \mathbb{Q}$ . Since  $\text{Rot}_{Y_1 \cap Y_\infty}^+(F)$  and  $\text{Rot}_{Y_1 \cap Y_\infty}^-(F)$  both contain  $\text{Rot}_{Y_1 \cap Y_\infty}(F)$  and are included in  $I$ , this gives (i).

Now we prove (v) for  $\min I$  (the case with the maximum is symmetric, and  $\partial I$  is reduced to two points). Suppose that  $\min \text{Rot}_{Y_1 \cap Y_\infty}(F) = p/q$  and let  $G = F^q - p$ . Then  $\min \text{Rot}_{Y_1 \cap Y_\infty}(G) = 0$ . Note that  $x \in Y_\infty + \mathbb{Z} \Rightarrow \forall n \geq 0$ ,  $G^n(x) \in Y + \mathbb{Z}$ . We apply Lemma 4.4 to the map  $G$  with  $Z = Y_1$  and  $\overline{T'}$ . Since (iii) is fulfilled, the set  $\overline{G(Z)} \cap Z$  is not empty. By refutation of Lemma 4.4, we get that there exist  $x \in \bigcup_{n \geq 0} \overline{G^n(Y_1) + \mathbb{Z}} \cap Y_\infty$  and  $n \geq 1$  such

that  $G^n(x) = x$ . Therefore,  $x$  is periodic (mod 1) for  $F$  and  $\rho_F(x) = p/q$ . Moreover,  $x \in Y_\infty$  according to the fact above. This gives (v) and concludes the proof of the proposition.  $\square$

**Example 4.7** The periodic (mod 1) point  $x$  given by Proposition 4.6(v) may not be in  $Y_1$ .

Let  $T$  be the universal covering of the graph sigma, and let  $F: T \rightarrow t$  be the continuous map of degree 1 such that  $F|_{\mathbb{R}} = Id$  and  $F$  is defined on the branch of  $T$  by:

- $F(a) = F(c) = a$ ,  $F(b) = e$ ,  $F(d) = a + 1$ ,  $F(e) = e + 1$ ,
- $F$  is affine on each of the intervals  $[a, b]$ ,  $[b, c]$ ,  $[c, d]$ ,  $[d, e]$ ,

where  $[a, e]$  is the branch of  $T$  with  $a \in \mathbb{R}$  and  $a < b < c < d < e$ . See Figure 3 for a picture of the map  $F$ . This entirely determines  $F$  because it is of degree 1.

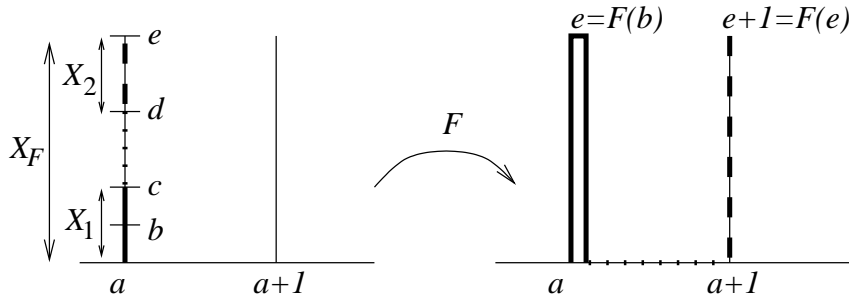


Figure 3: The action of  $F$  on the branch  $X_F$ .

$X_F$  is equal to  $[a, e]$  and the intervals given by Lemma 4.2 are  $X_1 = [a, c]$  (with  $p_1 = 0$ ) and  $X_2 = [d, e]$  (with  $p_2 = 1$ ).  $F$  is an affine Markov map and the restriction of its Markov graph to  $X_1, X_2$  is given in Figure 4. See [2, Section 6.1] for general results on Markov maps in this context, and in particular how it is possible to deduce periodic (mod 1) points and rotation numbers from the Markov graph.

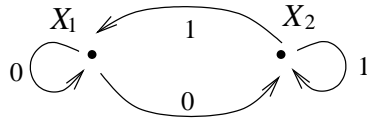


Figure 4: The Markov graph of  $F$  restricted to the vertices  $X_1$  and  $X_2$  (actually,  $X_1$  represents the two vertices  $[a, b]$  and  $[b, c]$ ). An arrow  $A \xrightarrow{i} B$  means that  $F(A) \supset B + i$ .

It can easily be deduced from the Markov graph of  $F$  that  $\text{Rot}_{X_1 \cap X_\infty}(F) = [0, 1]$  and the unique periodic (mod 1) point  $x \in X_F$  such that  $\rho_F(x) = 1$  is  $x = e$ , which does not belong to  $X_1$ .

In addition, we note that  $\text{Rot}_{\mathbb{R}}(F) = \{0\}$ . Thus  $\text{Rot}(F) = [0, 1]$  and  $\text{Rot}_{\mathbb{R}}(F)$  is not a connected component of  $\text{Rot}(F)$ .  $\square$

#### 4.4 Rotation set of $F$

Now, we are ready to prove that the set  $\text{Rot}(F)$  is closed and has finitely many connected components, and that every rational number in  $\text{Rot}(F)$  is the rotation number of some periodic (mod 1) point. Note that in the following theorem, the intervals  $I_0, \dots, I_k$  may be not disjoint; in particular  $I_0 = \text{Rot}_{\mathbb{R}}(F)$  may not be a connected component of  $\text{Rot}(F)$  (see Example 4.7).

**Theorem 4.8** *Let  $T \in \mathbf{T}^\circ$ ,  $F \in \mathcal{C}_1^\sigma(T)$  and  $X_\infty = \{x \in X \mid \forall n \geq 1, F^n(x) \in X + \mathbb{Z}\}$ . Then there exist an integer  $k \geq 0$  and compact non empty intervals  $I_0, \dots, I_k$  in  $\mathbb{R}$  such that:*

- $I_0 = \text{Rot}_{\mathbb{R}}(F) = \text{Rot}_{T_{\mathbb{R}}}(F)$ ,
- $\text{Rot}(F) = \text{Rot}^+(F) = \text{Rot}^-(F) = I_0 \cup \dots \cup I_k$ ,
- $\forall 1 \leq i \leq k, \forall r \in I_i \cap \mathbb{Q}$ , there exists a periodic (mod 1) point  $x \in X_\infty$  with  $\rho_F(x) = r$ ,
- $\forall 1 \leq i \leq k, I_i \cap \mathbb{Z} \neq \emptyset$ .

Moreover, if  $N$  is the integer given by Lemma 4.2, then  $k \leq N$ .

*Proof.* Consider  $X_1, \dots, X_N$  and  $p_1, \dots, p_N$  given by Lemma 4.2. We define inductively  $T_1, \dots, T_N \subset T$  and  $I_1, \dots, I_N \subset \mathbb{R}$  such that:

- (a)  $T_i$  is a connected subset of  $T$  such that  $T_i + 1 = T_i$ ,  $F(T_i) \subset T_i$  and, for all  $2 \leq i \leq N$ ,  $T_{i-1} \cup X_{i-1} \subset T_i$ .
- (b) either  $I_i$  is empty, or  $I_i$  is a compact interval containing  $p_i$  such that, for all  $r \in I_i \cap \mathbb{Q}$ , there exists a periodic (mod 1) point  $x \in X_\infty$  with  $\rho_F(x) = r$ .
- (c)  $\text{Rot}_{T_i \cup X_i}(F) = \text{Rot}_{T_i}(F) \cup I_i$ , and the same equality is valid with  $\text{Rot}^+$  and  $\text{Rot}^-$ .
- (d) If  $i \geq 2$ ,  $\text{Rot}_{T_i}(F) = \text{Rot}_{T_{i-1} \cup X_{i-1}}(F)$ , and the same equality is valid with  $\text{Rot}^+$  and  $\text{Rot}^-$ .

Let  $T_1 = T_{\mathbb{R}}$ . It satisfies (a). If  $X_1 \cap X_\infty = \emptyset$ , we take  $I_1 = \emptyset$ . Otherwise, we apply Proposition 4.6 with  $T' = T_1$ ,  $Y = X_F$  and  $X_1, \dots, X_N$  is place of  $Y_1, \dots, Y_M$ . It provides a compact interval  $I_1 = I = \text{Rot}_{X_1 \cap X_\infty}$  that satisfies (b). Moreover,  $\text{Rot}_{T_1 \cup X_1}(F) = \text{Rot}_{T_1}(F) \cup \text{Rot}_{X_1 \cap X_\infty}(F)$ , and the same equality is valid with  $\text{Rot}^+$  and  $\text{Rot}^-$ . Hence (c) is satisfied for  $i = 1$ .

Let  $i \geq 2$ . Suppose that  $T_j$  and  $I_j$  are already defined for all  $1 \leq j \leq i-1$ , and satisfy (a)-(d). Define

$$A_i = [\min X_F, \min X_{i-1}) \cup X_{i-1} \cup \left( \left( \bigcup_{n \geq 1} F^n(X_{i-1}) + \mathbb{Z} \right) \cap X_F \right).$$

For all  $n \geq 1$ ,  $F^n(\min X_{i-1}) \in T_{\mathbb{R}}$ , and thus  $(F^n(X_{i-1}) + \mathbb{Z}) \cap X_F$  is, either empty, or a compact subinterval of  $X_F$  containing  $\min X_F$ . Therefore,  $A_i$  is a subinterval of  $X$  containing  $\min X_F$  and  $X_{i-1}$ . Let  $T_i = T_{i-1} \cup (A_i + \mathbb{Z})$ . It is a connected subset of  $T$ ,  $T_i + 1 = T_i$  and  $T_{i-1} \cup X_{i-1} \subset T_i$ . Let us show that  $F(T_i) \subset T_i$ . Let  $x \in T_{i-1} \cup A_i$ . We distinguish 3 cases.

- If  $x \in T_{i-1}$  then  $F(x) \in T_{i-1}$  by invariance of  $T_{i-1}$ .
- If  $x \in [\min X_F, \min X_{i-1})$  then, either  $x \in X_1 \cup \dots \cup X_{i-2} \subset T_{i-1}$  and  $F(x) \in T_{i-1}$ , or  $x \in X \setminus (X_1 \cup \dots \cup X_N)$  and  $F(x) \in T_{\mathbb{R}}$ .
- If  $x \in \left( \bigcup_{n \geq 0} F^n(X_{i-1}) + \mathbb{Z} \right) \cap X_F$  then, either  $F(x) \in \left( \bigcup_{n \geq 0} F^n(X_{i-1}) + \mathbb{Z} \right) \cap (X_F + \mathbb{Z}) \subset A_i + \mathbb{Z}$ , or  $F(x) \in T_{\mathbb{R}}$ .

Consequently,  $F(T_i) \subset T_i$ , and (a) is satisfied. Moreover,

$$\text{Rot}_{T_i}(F) = \text{Rot}_{T_{i-1}}(F) \cup \text{Rot}_{X_{i-1}}(F) = \text{Rot}_{T_{i-1} \cup X_{i-1}}(F),$$

and the same equality is valid with  $\text{Rot}^+$  and  $\text{Rot}^-$ , which is (d) for  $i$ .

If  $F(X_i) \subset T_i$ , we take  $I_i = \emptyset$  and (b)-(c) are clearly satisfied. Otherwise, let  $b \in X_i$  such that  $F(b) \notin T_i$ . Let  $Y = \overline{T_i} \setminus \overline{T_i} \cap r^{-1}([0, 1))$  and  $Y_\infty = \{x \in Y \mid \forall n \geq 1, F^n(x) \in Y + \mathbb{Z}\}$ . The set  $Y$  is a compact subinterval of  $X_F$  and  $Y \cap \overline{T_i} = \{\min Y\}$ . We have  $b \in Y$  by invariance of  $T_i$ , and  $F(b) \in Y + p_i$  because  $F(X_i) \subset (X_F + p_i) \cup T_{\mathbb{R}}$ .

Let  $a = \max(\min X_i, \min Y)$ . We can define  $c = \min\{x \in [a, b] \mid F(x) \in Y + \mathbb{Z}\}$  because  $b \geq a$ . Moreover  $F(a) \in \overline{T_i}$  because  $F(\min X_i) \in T_{\mathbb{R}}$  and  $F(\min Y) \in F(\overline{T_i}) \subset \overline{T_i}$ . Therefore,  $F(c) = \min Y \pmod{1}$  by minimality. Let  $X'_i = [c, \max X_i] \subset X_i$ . We can apply Proposition 4.6 with  $T' = T_i$  and  $X'_i, X_{i+1}, \dots, X_N$  in place of  $Y_1, \dots, Y_M$ . We obtain a compact interval  $I_i = I = \text{Rot}_{X'_i \cap Y_\infty}(F)$  that satisfies (b) for  $i$ .

We have

$$X'_i \cap Y_\infty = \{x \in X_i \mid \forall n \geq 0, F^n(x) \notin \text{Int}(T_i)\}.$$

Therefore,  $\text{Rot}_{T_i \cup X_i}(F) = \text{Rot}_{T_i}(F) \cup \text{Rot}_{X'_i \cap Y_\infty}(F) = \text{Rot}_{T_i}(F) \cup I_i$ , and the same equality is valid with  $\text{Rot}^+$  and  $\text{Rot}^-$ . Hence (c) is satisfied for  $i$ . This concludes the construction of  $T_i$  and  $I_i$ .

Now, we end the proof of the theorem. Since  $F(T \setminus (T_N \cup X_N)) \subset T_N$ , it is clear that  $\text{Rot}(F) = \text{Rot}_{T_N \cup X_N}(F)$ . Combining this with (c) and (d), we get that

$$\begin{aligned} \text{Rot}(F) &= \text{Rot}_{T_N \cup X_N}(F) \\ &= \text{Rot}_{T_N}(F) \cup I_N \\ &= \text{Rot}_{T_{N-1} \cup X_{N-1}}(F) \cup I_N \\ &= \text{Rot}_{T_{N-1}}(F) \cup I_{N-1} \cup I_N \\ &= \dots \\ &= \text{Rot}_{T_1}(F) \cup I_1 \cup \dots \cup I_N \end{aligned}$$

and the same equalities are valid with  $\text{Rot}^+$  and  $\text{Rot}^-$ . Let  $I_0 = \text{Rot}_{\mathbb{R}}(F)$ . By Theorem 4.1,  $I_0$  is a non empty compact interval and  $I_0 = \text{Rot}_{T_{\mathbb{R}}}(F) = \text{Rot}_{T_{\mathbb{R}}}^+(F) = \text{Rot}_{T_{\mathbb{R}}}^-(F)$ . If  $x \in T$ , then either  $x \in (X_1 \cup \dots \cup X_N) + \mathbb{Z}$ , or  $F(x) \in T_1$ . Therefore,  $\text{Rot}_T(F) = \text{Rot}_{T_1 \cup X_1 \cup \dots \cup X_N}(F)$ , and the same equality is valid with  $\text{Rot}^+$  and  $\text{Rot}^-$ . To conclude, it remains to remove the empty intervals among  $I_1, \dots, I_N$ .  $\square$

If the empty rotation intervals are not removed in the proof of Theorem 4.8, then the theorem can be stated as follows:

**Theorem 4.8'** Let  $T \in \mathbf{T}^\circ$  and  $F \in \mathcal{C}_1^\sigma(T)$ . Let  $X_1, \dots, X_N, p_1, \dots, p_N$  be given by Lemma 4.2 and  $X_\infty = \{x \in X \mid \forall n \geq 1, F^n(x) \in X + \mathbb{Z}\}$ . Then there exist compact intervals  $I_0, \dots, I_N$  in  $\mathbb{R}$  such that:

- $I_0 = \text{Rot}_{\mathbb{R}}(F) = \text{Rot}_{T_{\mathbb{R}}}(F)$ ,
- $\text{Rot}(F) = \text{Rot}^+(F) = \text{Rot}^-(F) = I_0 \cup \dots \cup I_k$ ,
- for all  $1 \leq i \leq N$ , either  $I_i = \emptyset$  or  $p_i \in I_i$ ,
- $\forall 1 \leq i \leq N, \forall r \in I_i \cap \mathbb{Q}$ , there exists a periodic (mod 1) point  $x \in \overline{\bigcup_{n \geq 0} (F^n(X_1) + \mathbb{Z})} \cap X_\infty$  with  $\rho_F(x) = r$ ; if in addition  $r \in \text{Int}(I_i)$  then  $x$  can be chosen in  $X_1 \cap X_\infty$ .

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